

# SHOWING DISTINCTNESS OF SURFACE LINKS BY TAKING 2-DIMENSIONAL BRAIDS

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**ABSTRACT.** For an oriented surface link  $S$ , we can take a satellite construction called a 2-dimensional braid over  $S$ , which is a surface link in the form of a covering over  $S$ . We demonstrate that 2-dimensional braids over surface links are useful for showing the distinctness of surface links. We investigate non-trivial examples of surface links with free abelian group of rank two, concluding that their link types are infinitely many.

## 1. INTRODUCTION

A *surface link* is the image of a smooth embedding of a closed surface into Euclidean space  $\mathbb{R}^4$ . Two surface links are *equivalent* if there is an orientation-preserving self-diffeomorphism of  $\mathbb{R}^4$  carrying one to the other. In this paper, we assume that surface links are oriented. In [14], we investigated a satellite construction called a 2-dimensional braid over an oriented surface link, and introduced its graphical presentation called an  $m$ -chart on a surface diagram. A 2-dimensional braid over a surface link  $S$  is a surface link in the form of a covering over  $S$ , and can be regarded as an analog to a double of a classical link. One of expected applications of the notion of a 2-dimensional braid is that it will provide us with a method for showing the distinctness of surface links. The aim of this paper is to demonstrate such use for 2-dimensional braids.

Our main theorem is as follows. Let  $k$  be a positive integer. Let  $\sigma_1, \sigma_2, \dots, \sigma_k$  be the standard generators of the  $(k+1)$ -braid group. Let  $X_k = \sigma_1^2 \sigma_2 \sigma_3 \cdots \sigma_k$  where  $X_1 = \sigma_1^2$ , and let  $\Delta$  be a  $(k+1)$ -braid with a positive half twist. Let  $S_k = \mathcal{S}_{k+1}(X_k, \Delta^2)$ , a  $T^2$ -link called a torus-covering  $T^2$ -link determined from  $(k+1)$ -braids  $X_k$  and  $\Delta^2$ , and we take the first (respectively second) component of  $S_k$  as the one determined from the first (respectively second) strand of  $X_k$ ; see Section 2 for the construction. Here, a  $T^2$ -link is a surface link each of whose components is of genus one.

**Theorem 1.1.** *Abelian  $T^2$ -links of rank two,  $S_k$  and  $S_l$ , are not equivalent for distinct positive integers  $k$  and  $l$ . Thus, the link types of abelian  $T^2$ -links of rank two are infinitely many.*

An *abelian surface link* of rank  $n$  is a surface link whose link group is a free abelian group of rank  $n$  [5]; note that  $n$  is the number of the components. We remark that our abelian  $T^2$ -links of rank two cannot be distinguished

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by using link groups, and that by a homological argument we cannot show that their link types are infinitely many, but only that there are two link types; see Section 2.2. Our abelian  $T^2$ -link  $S_k$  of rank two is a sublink of the surface link given in [5], where we gave examples of abelian  $T^2$ -links of rank four, and we showed that their link types are infinitely many by calculations of triple linking numbers (see also Remark 2.3). Triple linking numbers are integer-valued invariants of surface links with at least three components; so we cannot use them for our case without a device. In order to overcome this situation, we take a 2-dimensional braid over  $S_k$  such that each component of  $S_k$  is split into two components. Then it has four components, and we can calculate triple linking numbers. A 2-dimensional braid over a surface link is obtained from the “standard” 2-dimensional braid by addition of braiding information. Unfortunately, if we consider the standard 2-dimensional braid, then the triple linking is trivial (Proposition 5.1). However, addition of braiding information makes a 2-dimensional braid with non-trivial triple linking, and enables us to show that  $S_k$  and  $S_l$  are not equivalent for distinct positive integers  $k$  and  $l$ . As a similar result, we refer to Suciu’s paper [20] where it is shown that there are infinitely many ribbon 2-knots in  $S^4$  with knot group the trefoil knot group.

The paper is organized as follows. In Section 2, we review torus-covering links and explain our example  $S_k$ , and we review triple linking numbers of torus-covering links. In Section 3, we review the notion of a 2-dimensional braid over a surface link. In Section 4, we review that a 2-dimensional braid of degree  $m$  over a surface link is presented by a finite graph called an  $m$ -chart on a surface diagram, and that 2-dimensional braids of degree  $m$  are equivalent if their surface diagrams with  $m$ -charts are related by local moves called Roseman moves. In Section 5, we show Proposition 5.1. In Section 6, we calculate triple linking numbers of a certain 2-dimensional braid over  $S_k$  and prove Theorem 1.1.

## 2. ABELIAN $T^2$ -LINKS OF RANK TWO

Our example  $S_k$  given in Theorem 1.1 is a surface link called a torus-covering link. In this section, we review torus-covering  $T^2$ -links; see [12] for details. We briefly observe that  $S_k$  is an abelian surface link of rank two, and that we cannot show that the link types of our examples are infinitely many by using a homological argument. Further, we review a formula for the triple linking numbers of torus-covering links [5].

**2.1. Torus-covering links.** Let  $T$  be a standard torus in  $\mathbb{R}^4$ , the boundary of an unknotted (standardly embedded) solid torus in  $\mathbb{R}^3 \times \{0\} \subset \mathbb{R}^4$ .

**Definition 2.1.** A *torus-covering  $T^2$ -link*  $S$  is a surface link in the form of a 2-dimensional braid over the standard torus  $T$ , i.e.  $S$  is a  $T^2$ -link in  $\mathbb{R}^4$  such that  $S$  is contained in a tubular neighborhood  $N(T)$  and  $\pi|_S : S \rightarrow T$  is an unbranched covering map, where  $\pi : N(T) \rightarrow T$  is the natural projection.

Let  $S$  be a torus-covering  $T^2$ -link. Fix a base point  $x_0 = (x'_0, x''_0)$  of  $T = S^1 \times S^1$ . Take two simple closed curves on  $T$ ,  $\mathbf{m} = \partial B^2 \times \{x''_0\}$  and  $\mathbf{l} = \{x'_0\} \times S^1$ . Recall that  $T$  is embedded as  $T = \partial(B^2 \times S^1) \subset \mathbb{R}^3 \times \{0\} \subset \mathbb{R}^4$ . Let us consider the intersections  $S \cap \pi^{-1}(\mathbf{m}) \subset B^2 \times \mathbf{m}$  and  $S \cap \pi^{-1}(\mathbf{l}) \subset$

$B^2 \times \mathbf{1}$ . They are regarded as closed  $m$ -braids in the 3-dimensional solid tori, where  $m$  is the degree of the covering map  $\pi|_S : S \rightarrow T$ . Cutting open the solid tori along the 2-disk  $\pi^{-1}(x_0) = B^2 \times \{x_0\}$ , we obtain two  $m$ -braids  $a$  and  $b$ . The assumption that  $\pi|_S$  is an unbranched covering implies that  $a$  and  $b$  commute. We call the commutative braids  $(a, b)$  the *basis braids* of  $S$ . Conversely, starting from a pair of commutative  $m$ -braids  $(a, b)$ , we can uniquely construct a torus-covering  $T^2$ -link with basis braids  $(a, b)$  [12, Lemma 2.8]. For commutative  $m$ -braids  $a$  and  $b$ , we denote by  $\mathcal{S}_m(a, b)$  the torus-covering  $T^2$ -link with basis braids  $(a, b)$ .

**2.2. Our abelian  $T^2$ -links of rank two.** We can check that our example  $S_k = \mathcal{S}_{k+1}(X_k, \Delta^2)$  is an abelian surface link, as follows. The link group of a torus-covering link  $\mathcal{S}_m(a, b)$  is a quotient group of the classical link group of the closure of  $a$  such that the abelianization is a free abelian group [12, Proposition 3.1]. Since the link group of the closure of  $X_k$ , a Hopf link, is a free abelian group of rank two, so is the link group of  $S_k$ .

We remark that by a homological argument we cannot show that our examples are infinitely many, but only that there are two link types. Let us consider the one-point compactification of  $\mathbb{R}^4$ , and regard that  $S_k$  is in the Euclidean 4-sphere  $S^4$ . Recall that we take the first (respectively second) component of  $S_k$  as the one determined from the first (respectively second) strand of  $X_k$ , and let us denote by  $F_1$  (respectively  $F_2$ ) the first (respectively second) component of  $S_k$ . Then, by Alexander's duality, we see that  $H_2(S^4 - F_1; \mathbb{Z}) \cong H_1(F_1; \mathbb{Z})$ , hence  $[F_2] = \mu + k\lambda \in H_2(S^4 - F_1; \mathbb{Z})$ , where  $(\mu, \lambda)$  is a preferred basis of  $H_1(F_1; \mathbb{Z}) \cong H_2(S^4 - F_1; \mathbb{Z})$  represented by a meridian and a preferred longitude of  $F_1$ . Similarly, let us denote by  $F'_1$  (respectively  $F'_2$ ) the first (respectively second) component of  $S_l$ . Then we can see that  $[F'_2] = \mu' + l\lambda' \in H_2(S^4 - F'_1; \mathbb{Z})$ , where  $(\mu', \lambda')$  is a preferred basis of  $H_1(F'_1; \mathbb{Z}) \cong H_2(S^4 - F'_1; \mathbb{Z})$  represented by a meridian and a preferred longitude of  $F'_1$ . Now, standardly embedded tori  $F_1$  and  $F'_1$  are related by an orientation-preserving self-diffeomorphism of  $S^4$  if and only if  $\begin{pmatrix} \mu' \\ \lambda' \end{pmatrix} = A \begin{pmatrix} \mu \\ \lambda \end{pmatrix}$  for  $A = \begin{pmatrix} \alpha & \beta \\ \delta & \gamma \end{pmatrix} \in GL_+(2; \mathbb{Z})$  such that  $\alpha + \beta + \gamma + \delta \equiv 0 \pmod{2}$  [11], which implies that  $[F_2] = [F'_2] \in H_2(S^4 - F_1; \mathbb{Z})$  if and only if  $k \equiv l \pmod{2}$  (see [6]).

**Remark 2.2.** The abelian surface link  $S_1$ , i.e.  $\mathcal{S}_2(\sigma_1^2, \sigma_1^2)$ , is the twisted Hopf 2-link we will mention in the proof of Proposition 5.1; see also [4].

**Remark 2.3.** It is known [9, Theorem 6.3.1–Exercise 6.3.3] that for classical links, the rank of an abelian link is at most two, and, for abelian links of rank two, there are exactly two link types; a positive Hopf link and a negative Hopf link.

**Remark 2.4.** Put  $T_m = \mathcal{S}_{k+1}(X_k, X_k^m)$  for an integer  $m$ . It is known ([1], see also [6, 12]) that  $T_m$  and  $T_n$  are equivalent for  $m \equiv n \pmod{2}$ . Fix the first component of  $T_m$  in the form of the standard torus. By a homological argument as in this section, we see that  $T_m$  cannot be taken to  $T_n$  for  $n \neq m$  by an orientation-preserving self-diffeomorphism of  $\mathbb{R}^4$  relative to the first component.

**2.3. Triple linking numbers of torus-covering links.** The triple linking number of a surface link  $S$  is defined as follows [2, Definition 9.1]. For the  $i$ th,  $j$ th, and  $k$ th components  $F_i, F_j, F_k$  of  $S$  with  $i \neq j$  and  $j \neq k$ , the *triple linking number*  $\text{Tlk}_{i,j,k}(S)$  of the  $i$ th,  $j$ th, and  $k$ th components of  $S$  is the total number of positive triple points minus the total number of negative triple points of a surface diagram of  $S$  such that the top, middle, and bottom sheet are from  $F_i, F_j$ , and  $F_k$ , respectively. Triple linking number is a link bordism invariant [3, 4, 18, 19]; for other properties, see [2, 3]. Triple linking numbers are useful for showing the distinctness of surface links with at least three components [5, 13, 15].

By [5], we have a formula for the triple linking numbers of a torus-covering  $T^2$ -link  $\mathcal{S}_m(a, b)$ .

We use the notations given in [5]. For a torus-covering  $T^2$ -link  $\mathcal{S}_m(a, b)$ , let  $A_i$  be the components of the closure of  $a$  which are from the  $i$ th component of  $\mathcal{S}_m(a, b)$ . Take one of the connected components of  $A_i$  and denote it by  $A_i^1$ . We define  $\text{lk}_{i,j}^a$  by the classical linking number

$$\text{lk}_{i,j}^a = \text{lk}(A_i^1, A_j),$$

where we regard  $A_i^1$  and  $A_j$  as oriented links in  $\mathbb{R}^3$ . The notation  $\text{lk}_{i,j}^b$  for the other basis braid is defined similarly. Note that  $\text{lk}_{i,j}^a$  does not depend on a choice of a connected component  $A_i^1$  [5, Remark 5.5], and note that  $\text{lk}_{i,j}^a$  is not always symmetric, i.e.  $\text{lk}_{i,j}^a$  is not always equal to  $\text{lk}_{j,i}^a$ .

For a torus-covering  $T^2$ -link, the triple linking number of the  $i$ th,  $j$ th and  $k$ th components is given by

$$(2.1) \quad \text{Tlk}_{i,j,k}(\mathcal{S}_m(a, b)) = -\text{lk}_{j,i}^a \text{lk}_{j,k}^b + \text{lk}_{j,k}^a \text{lk}_{j,i}^b,$$

where  $i \neq k$  and  $j \neq k$  [5, Theorem 5.4 and Remark 5.7].

### 3. TWO-DIMENSIONAL BRAIDS OVER A SURFACE LINK

A 2-dimensional braid, which is also called a simple braided surface, over a 2-disk, is an analogous notion of a classical braid [7, 8, 17]. We can modify this notion to a 2-dimensional braid over a closed surface [12], and further to a 2-dimensional braid over a surface link [3, Section 2.4.2], [14].

In this section, we review the notion of a 2-dimensional braid over a surface link [14].

**3.1. Two-dimensional braids over a surface link.** We use 2-dimensional braids without branch points over a closed surface, so our definition here is restricted to such surfaces; see [12, 14] for the definition which allows branch points.

Let  $\Sigma$  be a closed surface, let  $B^2$  be a 2-disk, and let  $m$  be a positive integer.

**Definition 3.1.** A closed surface  $\tilde{\Sigma}$  embedded in  $B^2 \times \Sigma$  is called a *2-dimensional braid over  $\Sigma$  of degree  $m$*  if the restriction  $\pi|_{\tilde{\Sigma}} : \tilde{\Sigma} \rightarrow \Sigma$  is an unbranched covering map of degree  $m$ , where  $\pi : B^2 \times \Sigma \rightarrow \Sigma$  is the natural projection.

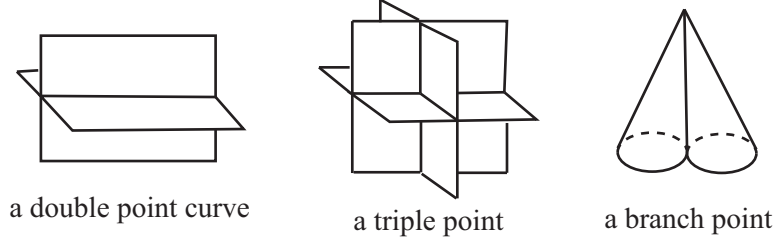


FIGURE 1. The singularity of a surface diagram.

Take a base point  $x_0$  of  $\Sigma$ . Two 2-dimensional braids over  $\Sigma$  of degree  $m$  are *equivalent* if there is a fiber-preserving ambient isotopy of  $B^2 \times \Sigma$  rel  $\pi^{-1}(x_0)$  which carries one to the other.

A surface link is said to be *of type*  $\Sigma$  when it is the image of an embedding of  $\Sigma$ . Let  $S$  be a surface link of type  $\Sigma$ , and let  $N(S)$  be a tubular neighborhood of  $S$  in  $\mathbb{R}^4$ .

**Definition 3.2.** A 2-dimensional braid  $\tilde{S}$  over  $S$  is the image of a 2-dimensional braid over  $\Sigma$  in  $B^2 \times \Sigma$  by an embedding  $B^2 \times \Sigma \rightarrow \mathbb{R}^4$  which identifies  $N(S)$  with  $B^2 \times \Sigma$  as a  $B^2$ -bundle over a surface. We define the *degree* of  $\tilde{S}$  as that of  $S$ .

Two 2-dimensional braids  $\tilde{S}$  and  $\tilde{S}'$  over surface links  $S$  and  $S'$  are *equivalent* if there is an ambient isotopy of  $\mathbb{R}^4$  carrying  $\tilde{S}$  to  $\tilde{S}'$  and  $N(S) = B^2 \times S$  to  $N(S') = B^2 \times S'$  as a  $B^2$ -bundle over a surface.

Equivalent 2-dimensional braids over surface links are also equivalent as surface links. A 2-dimensional braid  $\tilde{S}$  over  $S$  is a specific satellite with companion  $S$ ; see [3, Section 2.4.2], see also [10, Chapter 1].

**3.2. Standard 2-dimensional braids.** In this section, we define the standard 2-dimensional braid over a surface link  $S$ . Using this notion, we will explain in the next section that a 2-dimensional braid is presented by a finite graph called an  $m$ -chart on a surface diagram  $D$  of  $S$ . The standard 2-dimensional braid over  $S$  is the 2-dimensional braid presented by an empty  $m$ -chart on  $D$  [14].

First we will review a surface diagram of a surface link  $S$ ; see [3]. For a projection  $p : \mathbb{R}^4 \rightarrow \mathbb{R}^3$ , the closure of the self-intersection set of  $p(S)$  is called the singularity set. Let  $p$  be a generic projection, i.e. the singularity set of the image  $p(S)$  consists of double points, isolated triple points, and isolated branch points; see Figure 1. The closure of the singularity set forms a union of immersed arcs and loops, called double point curves. Triple points (respectively branch points) form the intersection points (respectively end points) of the double point curves. A *surface diagram* of  $S$  is the image  $p(S)$  equipped with over/under information along each double point curve with respect to the projection direction.

We define the  $2m$ -braid  $\tilde{\sigma}_1$  obtained from a 2-braid  $\sigma_1$ , as follows. For the proof of Theorem 1.1, here we define the  $mn$ -braid  $\tilde{b}$  obtained from an  $n$ -braid  $b$ . Let  $Q_m$  be  $m$  interior points of  $B^2$ . For a standard generator  $\sigma_i$  of an  $n$ -braid, let  $\tilde{\sigma}_i$  be the  $mn$ -braid obtained from  $\sigma_i$  in such a way that it

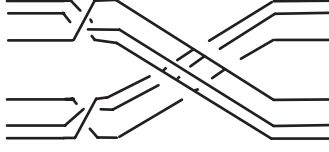


FIGURE 2. The  $2m$ -braid  $\widetilde{\sigma}_1$ .

is in the form of a  $Q_m$ -bundle over  $\sigma_i$  and it is obtained from  $\sigma_i$  by splitting each strand into a bundle of  $m$  parallel strands with a negative half twist at the initial points of each bundle; see Figure 2. The map taking  $\sigma_i$  to  $\widetilde{\sigma}_i$  determines a homomorphism from the  $n$ -braid group to the  $mn$ -braid group. For an  $n$ -braid  $b$ , let  $\widetilde{b}$  denote the image of  $b$  by this homomorphism.

**Definition 3.3.** Let  $S$  be a surface link. A surface diagram  $D$  of  $S$  consists of the following local parts: around (1) a regular point i.e. a nonsingular point, (2) a double point curve, (3) a triple point, and (4) a branch point. The case (1) is presented by an embedded 2-disk  $B^2$  with no singularity, and the case (2) is presented as the product of a 2-braid  $\sigma_1$  and an interval  $I$ .

We define the *standard 2-dimensional braid over  $S$*  locally for such local parts of  $D$  as follows: for (1), it is  $m$  parallel copies of  $B^2$ , and for (2), it is the product of the  $2m$ -braid  $\widetilde{\sigma}_1$  and  $I$ . Then, for the other cases (3) and (4), the standard 2-dimensional braid is naturally defined [14, Definition 5.1 and Proposition 5.2].

#### 4. CHART PRESENTATION OF 2-DIMENSIONAL BRAIDS AND ROSEMAN MOVES

In this section, we review the following. A 2-dimensional braid of degree  $m$  over a surface link  $S$  is presented by a finite graph called an  $m$ -chart on a surface diagram  $D$  of  $S$  [14]. For two 2-dimensional braids of degree  $m$ , they are equivalent if their surface diagrams with  $m$ -charts are related by a finite sequence of local moves called Roseman moves [14].

##### 4.1. Chart presentation of 2-dimensional braids over a surface link.

The graphical method called an  $m$ -chart on a 2-disk was introduced to present a simple surface braid which is a 2-dimensional braid over a 2-disk with trivial boundary condition [7, 8]. By regarding an  $m$ -chart on a 2-disk as drawn on a 2-sphere  $S^2$ , it presents a 2-dimensional braid over  $S^2$  [7, 8, 12]. This notion can be modified to an  $m$ -chart on a closed surface [12], and further to an  $m$ -chart on a surface diagram  $D$  of a surface link  $S$  [14]. A 2-dimensional braid over  $S$  is presented by an  $m$ -chart on  $D$  [14].

In this paper, we treat 2-charts with vertices of degree 2. We will just review the graphical form of an  $m$ -chart of a 2-dimensional braid over a surface link. See [14] for details.

Let  $\widetilde{S}$  be a 2-dimensional braid over a surface link  $S$ . Let  $D$  be a surface diagram of  $S$  by a projection  $p : \mathbb{R}^4 \rightarrow \mathbb{R}^3$  which is generic with respect to both  $S$  and  $\widetilde{S}$ . We can assume that the singularity set of the surface diagram

$p(\tilde{S})$  is the union of the singularity set of the diagram of the standard 2-dimensional braid over  $S$  and some finite graph  $\Gamma$  [14, Theorem 5.5]. Project  $\Gamma$  to  $D$  by the projection  $p(N(S)) = B^2 \times D \rightarrow D$ . Then we obtain a finite graph on the surface diagram  $D$ . An  $m$ -chart on a surface diagram  $D$  is such a finite graph equipped with certain additional information of orientations and labels assigned to the edges, where  $m$  is the degree of the 2-dimensional braid. Owing to the additional information, we can regain the original 2-dimensional braid from the  $m$ -chart on  $D$  [14] (see also [8]).

We can define an  $m$ -chart on  $D$  in graphical terms, where the labels of edges are from 1 to  $m-1$ ; see [14, Definitions 5.3 and 5.4]. Around a double point curve, an  $m$ -chart is as in Figure 3, with a vertex of degree 2. A 2-dimensional braid over  $S$  is presented by an  $m$ -chart on  $D$  [14, Theorem 5.5].

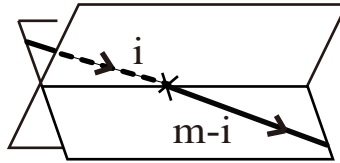


FIGURE 3. An  $m$ -chart around a double point curve, where  $i \in \{1, \dots, m-1\}$ . For simplicity, we omit the over/under information of each sheet.

**4.2. Roseman moves.** Roseman moves are local moves of surface diagrams as illustrated in Figure 4. It is known [16] that two surface links are equivalent if and only if their surface diagrams are related by a finite sequence of Roseman moves and ambient isotopies of the diagrams in  $\mathbb{R}^3$ . In [14], we introduced the notion of Roseman moves for surface diagrams with  $m$ -charts.

An  $m$ -chart is said to be *empty* if it is an empty graph.

**Definition 4.1.** We define *Roseman moves for surface diagrams with  $m$ -charts* by the local moves as illustrated in Figures 4 and 5, where we regard the diagrams in Figure 4 as equipped with empty  $m$ -charts.

Roseman moves for surface diagrams with  $m$ -charts as illustrated in Figures 4 and 5 are well-defined, i.e. for each pair of Roseman moves, the  $m$ -charts on the indicated diagrams present equivalent 2-dimensional braids [14, Theorem 6.2].

## 5. TRIPLE LINKING NUMBERS OF STANDARD 2-DIMENSIONAL BRAIDS

Recall the triple linking numbers (see Section 2.3). We will say that a surface link  $S$  has *trivial* triple linking if every triple linking number of  $S$  is zero or  $S$  consists of less than three components.

**Proposition 5.1.** *For the standard 2-dimensional braid  $\tilde{S}$  over a surface link  $S$ , if  $S$  has trivial triple linking, then so does  $\tilde{S}$ .*

*Proof.* Assume that  $S$  has trivial triple linking. It is known [4] that the link bordism class of a surface link is determined from triple linking numbers and another kind of link bordism invariants called double linking numbers, and

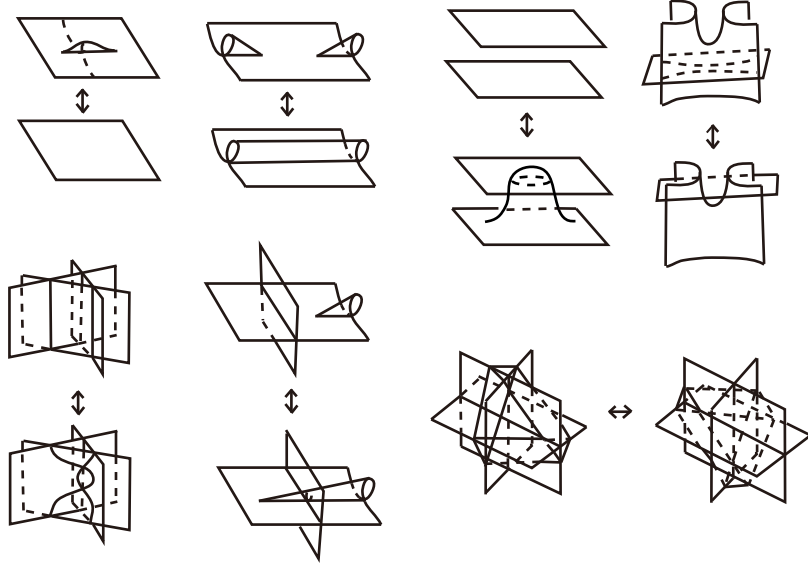


FIGURE 4. Roseman moves. For simplicity, we omit the over/under information of each sheet.

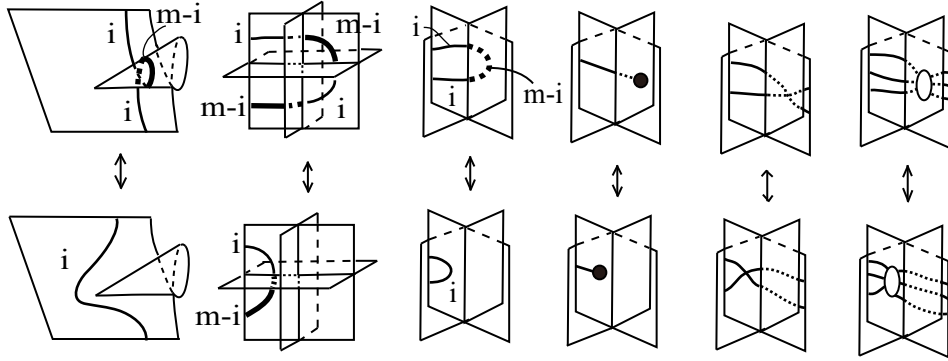


FIGURE 5. Roseman moves for surface diagrams with  $m$ -charts, where  $i \in \{1, \dots, m-1\}$ . For simplicity, we omit the over/under information of each sheet, and orientations and labels of edges of  $m$ -charts.

a surface link with trivial triple linking is link bordant to a split union of a finite number of trivial spheres and surface links called twisted Hopf 2-links, which has a surface diagram with no triple points (see also Remark 2.2). Hence  $S$  is link bordant to a surface link  $S'$  whose surface diagram has no triple points. By the well-definedness of Roseman moves,  $\tilde{S}$  is link bordant to the standard 2-dimensional braid  $\tilde{S}'$  over  $S'$ . Since the surface diagram of a standard 2-dimensional braid has triple points only around triple points of the companion surface [14], the surface diagram of  $\tilde{S}'$  has no triple points. Thus  $\tilde{S}$  is link bordant to a surface link with no triple points, which implies that  $\tilde{S}$  has trivial triple linking.  $\square$



## 6. PROOF OF THEOREM 1.1

In this section, we will consider a 2-dimensional braid  $\tilde{S}$  over a surface link  $S$  presented by a 2-chart consisting of a finite number of loops on a surface diagram of  $S$ . Here, a *loop* is a union of edges connected by vertices of degree 2 as in Figure 3. In our case of 2-charts, the edges are labeled by 1 and the orientations are coherent around a vertex of degree 2, so we can ignore the label information, and we regard the 2-chart on a surface diagram of  $S$  as oriented loops. Further, we consider that the loops are on  $S$  itself. By the well-definedness of Roseman moves, a 2-dimensional braid presented by a 2-chart  $\Gamma$  on  $S$  is equivalent to the 2-dimensional braid presented by a 2-chart  $f(\Gamma)$  on  $f(S)$  for an orientation-preserving self-diffeomorphism  $f$  of  $\mathbb{R}^4$ .

For a component  $F$  of a torus-covering  $T^2$ -link, we take a preferred basis of  $H_1(F; \mathbb{Z})$  represented by a pair of simple closed curves  $(\mu, \lambda)$  such that  $\mu$  (respectively  $\lambda$ ) is a connected component of  $F \cap \pi^{-1}(\mathbf{m})$  (respectively  $F \cap \pi^{-1}(\mathbf{l})$ ). Recall that  $\pi : N(T) \rightarrow T$  is the natural projection for a standard torus  $T$ , and  $\mathbf{m}$  and  $\mathbf{l}$  are simple closed curves on  $T$  given in Section 2.1. We will use the same notation  $(\mu, \lambda)$  for the preferred basis, and we call a simple closed curve in the homology class  $\mu$  (respectively  $\lambda$ ) a *meridian* (respectively a *preferred longitude*) of  $F$ . For a 2-chart  $\Gamma$  on  $F$  consisting of loops, we can assume that the intersections of the chart loops of  $\Gamma$  with a meridian  $\mu$  and a preferred longitude  $\lambda$  of  $F$  are transverse. We assign each intersection point the sign  $+1$  (respectively  $-1$ ) when it presents a positive (respectively negative) crossing, and we denote by  $I(\mu, \Gamma)$  (respectively  $I(\lambda, \Gamma)$ ) the sum of the signs of the intersection points of  $\Gamma$  with  $\mu$  (respectively  $\lambda$ ); note that we can assume that the chart loops are parallel by using local moves of charts called CI-moves of type (1) [8], and  $I(\mu, \Gamma)$  and  $I(\lambda, \Gamma)$  are well-defined for the homology classes  $\mu$  and  $\lambda$ .

For the torus-covering  $T^2$ -link  $S$  and its 2-dimensional braid  $\tilde{S}$  treated in this section, we take the first (respectively second) component of  $S$  as the one determined from the first (respectively second) strand of each basis braid of  $S$ , and we take the  $i$ th component of  $\tilde{S}$  as the one determined from the  $i$ th strand of each basis braid of  $\tilde{S}$  for  $i = 1, 2, 3, 4$ .

For the proof of Theorem 1.1, we calculate the triple linking numbers of a 2-dimensional braid of degree 2 over  $S_k$  in Theorem 1.1.

**Lemma 6.1.** *For the torus-covering  $T^2$ -link  $S_k$  for a positive integer  $k$  in Theorem 1.1, let us consider a 2-dimensional braid of degree 2 over  $S_k$ , denoted by  $\tilde{S}_k$ , which is presented by a 2-chart  $\Gamma$  consisting of loops on  $S_k$  such that it consists of 4 components. Then  $\text{Tlk}_{i,j,3}(\tilde{S}_k) = \text{Tlk}_{i,j,4}(\tilde{S}_k)$  for  $(i, j) = (1, 2)$  or  $(2, 1)$ , and  $\text{Tlk}_{i,j,1}(\tilde{S}_k) = \text{Tlk}_{i,j,2}(\tilde{S}_k)$  for  $(i, j) = (3, 4)$  or  $(4, 3)$ .*

*Proof.* The 2-dimensional braid  $\tilde{S}_k$  is also a torus-covering  $T^2$ -link. We denote by  $(a, b)$  the basis braids presenting  $\tilde{S}_k$ . Since  $\text{lk}_{j,3}^c = \text{lk}_{j,4}^c$  for  $j = 2, 1$ , and  $\text{lk}_{j,1}^c = \text{lk}_{j,2}^c$  for  $j = 4, 3$  ( $c = a, b$ ), by (2.1) we have the result.  $\square$

**Lemma 6.2.** *For the torus-covering  $T^2$ -link  $S_k$ , let us denote by  $F_1$  (respectively  $F_2$ ) the first (respectively second) component of  $S_k$ , and let  $(\mu_i, \lambda_i)$  be*

a preferred basis of  $H_1(F_i; \mathbb{Z})$  ( $i = 1, 2$ ). Let us consider a 2-dimensional braid  $\widetilde{S}_k$  as in Lemma 6.1, such that  $I(\mu_i, \Gamma) = 2p_i$  and  $I(\lambda_i, \Gamma) = 2q_i$ , for integers  $p_i$  and  $q_i$  ( $i = 1, 2$ ). Then  $\text{Tlk}_{1,2,3}(\widetilde{S}_k) = -kp_1 + q_1$  and  $\text{Tlk}_{2,3,4}(\widetilde{S}_k) = -p_2 + q_2$ .

Note that  $\widetilde{S}_k$  consists of 4 components if and only if  $I(\mu_i, \Gamma)$  and  $I(\lambda_i, \Gamma)$  ( $i = 1, 2$ ) are even, since these conditions are equivalent to the condition that  $\widetilde{S}_k \cap \pi_i^{-1}(\mu)$  and  $\widetilde{S}_k \cap \pi_i^{-1}(\lambda)$  ( $i = 1, 2$ ) are closed pure braids, where  $\pi_i : N(F_i) \rightarrow F_i$  is the natural projection.

*Proof.* The 2-dimensional braid  $\widetilde{S}_k$  is also a torus-covering  $T^2$ -link. We denote by  $(a, b)$  the basis braids presenting  $\widetilde{S}_k$ . We use the notation given in Section 3.2, taking  $m = 2$  and  $n = k + 1$ . Then,  $\text{lk}_{2,1}^a$  is determined from the linking number coming from the linking consisting of  $I(\mu_1, \Gamma)$  crossings and  $\widetilde{X}_k$ , that is,  $\text{lk}_{2,1}^a = p_1 + \text{lk}_{2,1}^{\widetilde{X}_k}$ , and similarly,  $\text{lk}_{2,1}^b = q_1 + \text{lk}_{2,1}^{\Delta^2}$ . By definition, for a braid  $c$ , the braid  $\widetilde{c}$  has a negative (respectively positive) half twist at the place which is a fiber of a point of each arc forming a positive (respectively negative) crossing of  $c$ ; hence,  $\text{lk}_{2,1}^{\widetilde{X}_k} = -\text{lk}_{1,2}^{X_k}$  and  $\text{lk}_{2,1}^{\Delta^2} = -\text{lk}_{1,2}^{\Delta^2}$ . thus  $\text{lk}_{2,1}^a = p_1 - \text{lk}_{1,2}^{X_k}$  and  $\text{lk}_{2,1}^b = q_1 - \text{lk}_{1,2}^{\Delta^2}$ .

Further,  $\text{lk}_{2,3}^a = \text{lk}_{1,2}^{X_k}$  and  $\text{lk}_{2,3}^b = \text{lk}_{1,2}^{\Delta^2}$ . Thus  $\text{Tlk}_{1,2,3}(\widetilde{S}_k) = -p_1 \text{lk}_{1,2}^{\Delta^2} + q_1 \text{lk}_{1,2}^{X_k}$  by (2.1). Since  $\text{lk}_{1,2}^{X_k}$  is the linking number of the closure of  $X_k$ ,  $\text{lk}_{1,2}^{X_k} = 1$ . Since  $F_1$  (respectively  $F_2$ ) is constructed by one strand (respectively  $k$  strands) of  $\Delta^2$ ,  $\text{lk}_{1,2}^{\Delta^2} = k$ . Thus  $\text{Tlk}_{1,2,3}(\widetilde{S}_k) = -kp_1 + q_1$ .

By the same argument, we have  $\text{Tlk}_{2,3,4}(\widetilde{S}_k) = -p_2 \text{lk}_{2,1}^{\Delta^2} + q_2 \text{lk}_{2,1}^{X_k}$  by (2.1), and  $\text{lk}_{2,1}^{X_k} = 1$ . Since  $\Delta^2$  is a pure braid, we see that  $\text{lk}_{2,1}^{\Delta^2} = 1$ . Thus  $\text{Tlk}_{2,3,4}(\widetilde{S}_k) = -p_2 + q_2$ .  $\square$

*Proof of Theorem 1.1.* Let  $k$  and  $l$  be positive integers. We denote by  $F_1$  (respectively  $F_2$ ) the first (respectively second) component of  $S_k$ , and we denote by  $F'_1$  (respectively  $F'_2$ ) the first (respectively second) component of  $S_l$ .

First we show that for  $k \neq l$ , there does not exist an orientation-preserving self-diffeomorphism of  $\mathbb{R}^4$  carrying  $F_1$  to  $F'_1$  and  $F_2$  to  $F'_2$ . Assume that there is such a diffeomorphism  $f$ . Let us consider a 2-dimensional braid over  $S_k$ , denoted by  $\widetilde{S}_k^1$ , which is presented by a 2-chart  $\Gamma$  on  $S_k$  such that  $\Gamma \cap F_1$  consists of loops with  $I(\mu_1, \Gamma) = 2p$  and  $I(\lambda_1, \Gamma) = 2q$  and  $\Gamma \cap F_2 = \emptyset$ , where  $(\mu_1, \lambda_1)$  is a preferred basis of  $H_1(F_1; \mathbb{Z})$ . Note that  $\widetilde{S}_k^1$  consists of 4 components.

Since  $f$  is an orientation-preserving diffeomorphism which carries  $F_1$  to  $F'_1$ ,  $f|_{F_1}$  is an orientation-preserving diffeomorphism from a torus  $F_1$  to a torus  $F'_1$ . Let  $A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \text{GL}_+(2, \mathbb{Z})$  be a matrix determined by

$$(6.1) \quad \begin{pmatrix} \mu'_1 \\ \lambda'_1 \end{pmatrix} = A \begin{pmatrix} f_*(\mu_1) \\ f_*(\lambda_1) \end{pmatrix},$$

where  $(\mu'_1, \lambda'_1)$  is a preferred basis of  $H_1(F'_1; \mathbb{Z})$ .

Put  $\Gamma' = f(\Gamma)$ . By  $f$ ,  $\widetilde{S}_k^1$  is taken to a 2-dimensional braid over  $S_l$ , presented by a 2-chart  $\Gamma'$  on  $S_l$  such that  $\Gamma' \cap F_1'$  consists of loops and  $\Gamma' \cap F_2' = \emptyset$ , which will be denoted by  $\widetilde{S}_l^1$ . We see that  $I(f_*(\mu_1), \Gamma') = I(\mu_1, \Gamma) = 2p$ , and  $I(f_*(\lambda_1), \Gamma') = I(\lambda_1, \Gamma) = 2q$ . Put  $p' = I(\mu_1', \Gamma')/2$  and  $q' = I(\lambda_1', \Gamma')/2$ ; note that  $p'$  and  $q'$  are integers, since  $\widetilde{S}_l^1$  consists of 4 components. It follows from (6.1) that

$$(6.2) \quad \begin{pmatrix} p' \\ q' \end{pmatrix} = A \begin{pmatrix} p \\ q \end{pmatrix}.$$

Since the triple linking numbers  $\text{Trk}_{1,2,3}$  for  $\widetilde{S}_k^1$  and  $\widetilde{S}_l^1$  are the same, Lemma 6.2 implies that

$$(6.3) \quad -kp + q = -lp' + q',$$

hence, it follows from (6.2) that  $kp - q = (\alpha l - \gamma)p + (\beta l - \delta)q$ . Since this equation holds true for any integers  $p$  and  $q$ ,

$$(6.4) \quad \begin{pmatrix} k \\ -1 \end{pmatrix} = A^T \begin{pmatrix} l \\ -1 \end{pmatrix},$$

where  $A^T$  is the transposed matrix of  $A$ .

Next we will consider another 2-dimensional braid over  $S_k$ , denoted by  $\widetilde{S}_k^2$ , presented by a 2-chart  $\widetilde{\Gamma}$  on  $S_k$  such that  $\widetilde{\Gamma} \cap F_1 = \emptyset$  and  $\widetilde{\Gamma} \cap F_2$  consists of loops on  $F_2$  and moreover  $\widetilde{\Gamma} \cap F_2$  is the preimage by the projection  $N(T) \rightarrow T$  of a 2-chart  $\Gamma$  on the standard torus  $T$  consisting of loops with  $I(\mathbf{m}, \Gamma) = 2p$  and  $I(\mathbf{l}, \Gamma) = 2q$ , where  $(\mathbf{m}, \mathbf{l})$  is a preferred basis of  $T$ . Note that  $I(\mu_2, \widetilde{\Gamma}) = 2kp$  and  $I(\lambda_2, \widetilde{\Gamma}) = 2q$ , where  $(\mu_2, \lambda_2)$  is a preferred basis of  $H_1(F_2; \mathbb{Z})$ .

Let  $g$  be an orientation-preserving diffeomorphism of  $\mathbb{R}^4$  which carries  $F_2$  sufficiently close to  $F_1$  and  $(g|_{F_i})_* = \text{id}_* : H_1(F_i; \mathbb{Z}) \rightarrow g_*(H_1(F_i; \mathbb{Z}))$  ( $i = 1, 2$ ). Further, we assume that  $T$  is sufficiently close to  $F_1$ . Then  $\begin{pmatrix} \mathbf{m}' \\ \mathbf{l}' \end{pmatrix} = A \begin{pmatrix} (f \circ g)_*(\mathbf{m}) \\ (f \circ g)_*(\mathbf{l}) \end{pmatrix}$ , where  $(\mathbf{m}', \mathbf{l}')$  is a preferred basis of  $T' = (f \circ g)(T)$ . Put  $\Gamma' = (f \circ g)(\Gamma)$ . Then we have

$$(6.5) \quad \begin{pmatrix} I(\mathbf{m}', \Gamma') \\ I(\mathbf{l}', \Gamma') \end{pmatrix} = A \begin{pmatrix} I(\mathbf{m}, \Gamma) \\ I(\mathbf{l}, \Gamma) \end{pmatrix}.$$

Put  $S' = (f \circ g)(S_k)$ . The surface link  $S'$  is in the form of a 2-dimensional braid over  $T'$  of degree  $k + 1$ . For the natural projection  $\pi' : N(T') = (f \circ g)(N(T)) \rightarrow T'$  and a meridian  $\mathbf{m}'$  and a preferred longitude  $\mathbf{l}'$  of  $T'$ , let us consider  $S' \cap \pi'^{-1}(\mathbf{m}')$  and  $S' \cap \pi'^{-1}(\mathbf{l}')$ , which are closed  $(k + 1)$ -braids in the 3-dimensional solid tori. In the same way of obtaining basis braids, we obtain  $(k + 1)$ -braids from the closed braids by cutting open the solid tori along the 2-disk  $\pi'^{-1}(x'_0)$ , where  $x'_0$  is the intersection point of  $\mathbf{m}'$  and  $\mathbf{l}'$ . We denote the braids by  $a$  and  $b$ . Note that here  $T'$  is a standard torus, and hence  $(a, b)$  are basis braids, but we can apply the same argument if  $T'$  is not a standard torus. Since  $S'$  consists of two components,  $a$  and  $b$  satisfy one of the three cases as follows.

(Case 1) The closure of  $a$  is a link consisting of two components, and  $b$  is a pure braid.

- (Case 2) Each of the closures of  $a$  and  $b$  is a link consisting of two components.  
 (Case 3) The braid  $a$  is a pure braid, and the closure of  $b$  is a link consisting of two components.

Put  $\tilde{\Gamma}' = (f \circ g)(\tilde{\Gamma})$ . By  $f \circ g$ ,  $\tilde{S}_k^2$  is taken to a 2-dimensional braid presented by a 2-chart  $\tilde{\Gamma}'$  on  $S'$ , which will be denoted by  $\tilde{S}'$ . We denote by  $F'$  the component  $(f \circ g)(F_2)$  of  $S'$ , and we denote by  $(\mu', \lambda')$  a preferred basis of  $H_1(F'; \mathbb{Z})$ . Since  $\tilde{\Gamma} \cap F_2$  is in the form of the preimage by  $N(T) \rightarrow T$  of the 2-chart  $\Gamma$  on  $T$ ,  $\tilde{\Gamma}' \cap F'$  is in the form of the preimage by  $N(T') \rightarrow T'$  of the 2-chart  $\Gamma'$  on  $T'$ , and hence  $I(\mu', \tilde{\Gamma}') = i \cdot I(\mathbf{m}', \Gamma')$  and  $I(\lambda', \tilde{\Gamma}') = j \cdot I(\mathbf{l}', \Gamma')$  for  $(i, j) = (k, 1)$  for Case 1,  $(k, k)$  for Case 2, and  $(1, k)$  for Case 3. Thus

$$(6.6) \quad \begin{pmatrix} I(\mu', \tilde{\Gamma}') \\ I(\lambda', \tilde{\Gamma}') \end{pmatrix} = B \begin{pmatrix} I(\mathbf{m}', \Gamma') \\ I(\mathbf{l}', \Gamma') \end{pmatrix},$$

where  $B$  is a diagonal matrix  $\text{diag}(i, j)$  such that  $(i, j) = (k, 1)$  for Case 1,  $(k, k)$  for Case 2, and  $(1, k)$  for Case 3.

Put  $h = f \circ (f \circ g)^{-1}$ . Then  $h$  is an orientation-preserving self-diffeomorphism of  $\mathbb{R}^4$  which carries  $S'$  to  $S_l$ . In particular,  $h$  carries  $F'$  to the second component  $F'_2$  of  $S_l$ . Let  $C = \begin{pmatrix} \alpha' & \beta' \\ \gamma' & \delta' \end{pmatrix} \in \text{GL}_+(2, \mathbb{Z})$  be a matrix determined by  $\begin{pmatrix} \mu'_2 \\ \lambda'_2 \end{pmatrix} = C \begin{pmatrix} h_*(\mu') \\ h_*(\lambda') \end{pmatrix}$ , where  $(\mu'_2, \lambda'_2)$  is a preferred basis of  $H_1(F'_2; \mathbb{Z})$ . Put  $\Gamma'' = h(\tilde{\Gamma}')$ . Then

$$(6.7) \quad \begin{pmatrix} I(\mu'_2, \Gamma'') \\ I(\lambda'_2, \Gamma'') \end{pmatrix} = C \begin{pmatrix} I(\mu', \tilde{\Gamma}') \\ I(\lambda', \tilde{\Gamma}') \end{pmatrix}.$$

Put  $p'' = I(\mu'_2, \Gamma'')/2$  and  $q'' = I(\lambda'_2, \Gamma'')/2$ , which are integers. Since  $I(\mathbf{m}, \Gamma) = 2p$  and  $I(\mathbf{l}, \Gamma) = 2q$ , together with (6.5)–(6.7), we have

$$(6.8) \quad \begin{pmatrix} p'' \\ q'' \end{pmatrix} = (CBA) \begin{pmatrix} p \\ q \end{pmatrix}.$$

By the composite diffeomorphism  $h \circ f \circ g = f$ ,  $\tilde{S}_k^2$  is taken to a 2-dimensional braid over  $S_l$ , which will be denoted by  $\tilde{S}_l^2$ . Since  $\text{Tk}_{2,3,4}$  are the same for  $\tilde{S}_k^2$  and  $\tilde{S}_l^2$ , together with  $I(\mu_2, \tilde{\Gamma}) = 2kp$  and  $I(\lambda_2, \tilde{\Gamma}) = 2q$ , Lemma 6.2 implies that

$$(6.9) \quad -kp + q = -p'' + q''.$$

Since this equation holds true for any integers  $p$  and  $q$ , it follows from (6.8) that  $\begin{pmatrix} k \\ -1 \end{pmatrix} = (CBA)^T \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ . Thus, together with (6.4),  $B^T C^T \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} l \\ -1 \end{pmatrix}$ , hence  $i(\alpha' - \gamma') = l$  and  $j(\beta' - \delta') = -1$ . Let us assume  $k > l > 0$ . For Cases 1 and 2,  $k(\alpha' - \gamma') = l$  from the first equation. This contradicts the assumption that  $k > l > 0$ . For Case 3, the second equation implies that  $k(\delta' - \beta') = 1$ , which contradicts the assumption that  $k > 1$ . Thus, for  $k \neq l$ , there does not exist an orientation-preserving self-diffeomorphism of  $\mathbb{R}^4$  which carries  $F_1$  to  $F'_1$  and  $F_2$  to  $F'_2$ .

Next we show that for  $k \neq l$ , there does not exist an orientation-preserving self-diffeomorphism of  $R^4$  which carries  $F_1$  to  $F'_2$  and  $F_2$  to  $F'_1$ . We will discuss a similar argument as in the former case of a diffeomorphism which carries  $F_1$  to  $F'_1$  and  $F_2$  to  $F'_2$ , using the same notation except where we give notice.

Assume that there is such a diffeomorphism  $f$ , and consider  $\Gamma$  as in the former case. Then, since  $\text{Tk}_{1,2,3}$  for  $\widetilde{S}_k^1$  and  $\text{Tk}_{3,4,1} = \text{Tk}_{4,3,2}$  (see Lemma 6.1) for  $\widetilde{S}_l^1$  are the same, and since  $\text{Tk}_{4,3,2} = -\text{Tk}_{2,3,4}$  [2], Lemma 6.2 implies that instead of (6.3) we have

$$(6.10) \quad -kp + q = p' - q',$$

where  $p' = I(\mu'_2, \Gamma')/2$  and  $q' = I(\lambda'_2, \Gamma')/2$ , and hence instead of (6.4) we have

$$(6.11) \quad \begin{pmatrix} k \\ -1 \end{pmatrix} = A^T \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

Next we will consider another 2-dimensional braid over  $S_k$ , denoted by  $\widetilde{S}_k^2$ , presented by the 2-chart  $\widetilde{\Gamma}$  as in the former case. Then, by the same argument as in the former case, we have (6.8), where  $p'' = I(\mu'_1, \Gamma'')/2$  and  $q'' = I(\lambda'_1, \Gamma'')/2$ .

By the composite diffeomorphism  $h \circ f \circ g$ ,  $\widetilde{S}_k^2$  is carried to a 2-dimensional braid over  $S_l$ , which will be denoted by  $\widetilde{S}_l^2$ . Since  $\text{Tk}_{2,3,4}$  for  $\widetilde{S}_k^2$  and  $\text{Tk}_{3,1,2} = \text{Tk}_{3,2,1}$  (see Lemma 6.1) for  $\widetilde{S}_l^2$  are the same, and since  $\text{Tk}_{3,2,1} = -\text{Tk}_{1,2,3}$  [2], together with  $I(\mu_2, \widetilde{\Gamma}) = 2kp$  and  $I(\lambda_2, \widetilde{\Gamma}) = 2q$ , Lemma 6.2 implies that

$$(6.12) \quad -kp + q = lp'' - q''.$$

Since this equation holds true for any integers  $p$  and  $q$ , it follows from (6.8) that  $\begin{pmatrix} k \\ -1 \end{pmatrix} = (CBA)^T \begin{pmatrix} -l \\ 1 \end{pmatrix}$ . Thus, together with (6.11),  $B^T C^T \begin{pmatrix} -l \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ , hence  $i(-l\alpha' + \gamma') = -1$  and  $j(-l\beta' + \delta') = 1$ . Let us assume  $k > l > 0$ . Since at least one of  $i$  and  $j$  is  $k$  for Cases 1, 2, and 3, these equations contradict the assumption that  $k > 1$ . Thus, for  $k \neq l$ , there does not exist an orientation-preserving self-diffeomorphism of  $R^4$  carries  $F_1$  to  $F'_2$  and  $F_2$  to  $F'_1$ . Thus  $S_k$  and  $S_l$  are not equivalent for positive integers  $k \neq l$ . □

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